Math 210A Lecture 24 Notes

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1 Artinian and Noetherian Rings

1.1 Maximal ideals

Theorem 1.1. *Let I be an ideal of a ring R. Then there exists a maximal ideal of R containing I.*

Proof. Let X be the set of proper ideals of R containing R. If C is a chain in X, $N = \bigcup_{J \in C} J$ is and ideal containing I, and $1 \notin N$, so $N \neq R$. So $\mathbb{C}$ has an upper bound. By Zorn's lemma, X has a maximal element, which is a maximal ideal containing I. $\square$

Proposition 1.1. *Maximal ideals in a commutative ring are prime.*

Proof. We have already proved that m is maximal iff R/m is a simple ring and that in a commutative ring, p is prime iff R/p is an integral domain. If R is commutative, then R/m is a division ring. $\square$

Remark 1.1. (0) is prime iff R is a domain.

Example 1.1. $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$, and $\mathbb{Z}[x]/(p) \cong \mathbb{F}_p[x]$.

1.2 Artinian and noetherian rings

Definition 1.1. Let $(I, \leq)$ be a partially ordered set. A chain $a_1 \leq a_2 \leq a_3 \leq \cdots$ satisfies the **ascending chain condition (ACC)** if there exists some N such that $a_k = a_N$ for all $k \geq N$. A chain $a_1 \geq a_2 \geq a_3 \geq \cdots$ satisfies the **descending chain condition (DCC)** if there exists some N such that $a_k = a_N$ for all $k \geq N$.

Definition 1.2. An R-module is **noetherian** if its set of R-submodules satisfies the ACC. And R module is **artinian** if its R submodules satisfy the DCC.[1]

[1] Noetherian and artinian are words used so commonly that they are often not capitalized, like abelian.

Definition 1.3. A ring is **left noetherian** (resp. **left artinian**) if it is noetherian (resp. artinian) as a left module over itself. A ring is **noetherian** (resp. **artinian**) if it is left and right noetherian (resp. artinian).

Example 1.2. The polynomial ring $F[x_1, x_2, x_3, \dots]$ is not noetherian. It has the infinite ascending chain

$$0 \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \cdots .$$

Example 1.3. $F[x]/(x^n)$ is both artinian and noetherian. Check that all ideals of this ring have the form (x^i) for $0 \leq i \leq n$.

Proposition 1.2. *Finite products of division rings are artinian and noetherian.*

Proposition 1.3. *An R-module M is noetherian iff every submodule of M is finitely generated.*

Proof. ($\Longleftarrow$): Suppose $(N_i)_{i=1}^{\infty}$ is anascending chain of R-sbumodules of M. Then $N = \bigcup_{i=1}^{\infty} N_i$ is an R-submodule of M. Then N is gnerated by $m_1, \dots, m_k \in N$. Each $m_i \in N_{j_i}$ for some $j_i \geq 1$. Every m_i is in $N_{\max j_i}$. So $N_{\max j_i} = N$.

($\Longrightarrow$): Let M be noetherian, and let $N \subseteq M$ be a submodule. If $N \neq 0$, then take $a_1 \in N \setminus (0)$. Set $N_1 = Ra_1$. If possible, take $a_i \in N \setminus N_i$, and set $N_{i+1} = N_i + Ra_{i+1} = R(a_1, \dots, a_{i+1})$. Then

$$(0) = N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots ,$$

so this pocess must terminate; i.e. there exists some i such that $N_i = N$, and N_i is finitely generated. $\square$

Corollary 1.1. *PIDs are noetherian.*

Example 1.4. $F[x]$ is noetherian.

Proposition 1.4. *Let M be an R-module and N be an R-submodule of M. THen M is noetherian iff N and M/N are noetherian.*

Proof. ($\Longrightarrow$): If N is noetherian, then submodules of M are finitely generated. Then submodules of N are finitely generated, so N is Noetherian. Now let $A \subseteq M/N$ is an R-submodule and $\pi L M \to M/N$ be the quotient map. Then $\pi^{-1}(A)$ is finitely generated and π applied to the generators generate A.

($\Longleftarrow$): Let $P \subseteq M$ be an R submodule. Then $P \cap N \subseteq N$ and $(P+N)/N \subseteq M/N$ are submodules of N and M/N, so they are finitely generated. Note that $(P+N)/N \cong P/(P \cap N)$. If $p_1, \dots, p_k$ generated $P \cap N$ and $q_1, \dots, q_\ell$ generate $P/(P \cap N)$, then we claim that $p_1, \dots, p_k, q_1' \in \pi_P^{-1}(\{q_1\}), \dots, q_\ell' \in \pi_P^{-1}(\{q_\ell\})$ generate P, where $\pi_P : P \to P/(P \cap N)$. If $a \in P$, then $\pi_P(a) = \sum_{i=1}^{\ell} r_i q_i$ for $r_i \in R$, and then $a - \sum_{i=1}^{\ell} r_i q_i' \in P \cap N$. So it equals $\sum_{j=1}^{k} s_j p_j$, where $s_-j \in R$. $\square$

Corollary 1.2. *If R is noetherian, then R^n is noetherian for $n \in N^+$.*

Proof. Induct on n. The inductive step follows form $R^{n+1}/R \cong R^n$. $\square$

Proposition 1.5. *Every finitely generated module over a left noetherian ring is noetherian.*

Proof. Let M be a finitely generated R-module, where R is left-noetherian, and let the finite list of generators be $a_1, \dots, a_n \in M$. R^n is a free R-module of rank n, so there exists a unique $\phi : R^n \to M$ such that $\phi(e_i) = a_i$ for all i. Then ϕ is onto. Let $N \subseteq M$ be a submodule, and consider the R-submodule $N' = \phi^{-1}(N) \subseteq R^n$. R^n is noetherian, so since N' is finitely generated, N is finitely generated. $\square$

Definition 1.4. A domain R is a **unique factorization domain (UFD)** if every element $a \in R \setminus \{0\}$ can be written as $a = u\pi_1 \cdots \pi_k$ with $u \in R^\times$, $\pi_i \in R$ irreducible, and if $a = vp_1, \dots p_\ell$ with $v \in R^\times$ and $p_i \in R$ irreducible, then $k = \ell$ and there exists a permutation $\sigma \in S_k$ such that $\pi \sim p_{\sigma(i)}$ for all i.